

# Criticality in transport through the quantum Ising chain

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We consider thermal transport between two reservoirs coupled by a quantum Ising chain as a model for non-equilibrium physics induced in quantum-critical many-body systems. By deriving rate equations based on exact expressions for the quasiparticle pairs generated during the transport, we observe signatures of the underlying quantum phase transition in the steady-state energy current already at finite reservoir temperatures.

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Quantum phase transitions (QPTs) are drastic manifestations of quantum fluctuations in many-body systems that lead to critical behavior and states of matter with symmetry-broken phases [1]. Traditionally, such changes of state are associated with changes in ground state properties at zero temperature. Recently, considerable attention has turned to the role of excited states, as these become important if the fate of QPTs under modifications such as finite temperature or non-equilibrium due to dissipation, external control, or driving is addressed. Among the fundamental issues then is the very definition of a QPT under non-equilibrium conditions, and furthermore the question of how non-analyticities of the ground state, phase diagrams, critical exponents, scaling behavior, or the order of the transition are modified.

From the point of view of recent advances in simulations and realizations of QPTs in, e.g., cold atoms [2, 3], these are urgent issues, as experimental systems always contain a certain amount of dissipation and require external control and read-out mechanisms. For example, a classical external control allows one to explore novel states of matter and effective interactions which are absent in equilibrium [4–7] and lead to new types of phase diagrams with additional phases, transition lines and critical points. Similar results have been obtained for QPTs in open, dissipative systems, either described by additional dissipative terms in equations of motions (similar to mean-field-type laser equations [8, 9]) or Lindblad-type master equations, e.g. [10]. A further line of theoretical work concerning excited-state phase transitions has also mostly been restricted to mean-field type QPTs (such as the Lipkin-Meshkov-Glick [11] or the Dicke-superradiance [12] model), where non-analyticities in quantities like the density of states or excited state energies are essentially related to singularities in a (classical) energy landscape. At zero system temperature, these can be made visible by suitable measurement devices – as has e.g. been suggested for the Dicke model [13].

In this letter, we investigate whether such features are also accessible in the non-equilibrium regime at the example of the quantum Ising chain in a transverse magnetic field. Its interesting properties can either be studied in existing Ising ferromagnets [14] or in corresponding

quantum simulators, e.g. based on NMR techniques [15], trapped ions [16], atomic spins [17, 18], or electrons floating on liquid helium [19]. Quite some work has been devoted to thermal transport along open spin chains [21–24]. Complementing these approaches, we consider here a scenario where transport occurs perpendicularly through a closed ring of spins. Going beyond linear response [25], this enables us to explore the extreme non-equilibrium regime. We use the chain, which has a transition from a paramagnetic into a  $Z_2$  symmetry-broken ferromagnetic phase with long-range magnetic ordering at zero temperature, as a thermal coupling between two reservoirs whose temperature gradient drives a stationary, thermal current. Combining exact results for the Ising model with the weak-coupling, Born-Markov-Secular type description of open systems [20], we demonstrate that the thermal current displays criticality at the same value of the Ising chain coupling parameter as for the zero temperature (ground state) QPT. This has to be contrasted with stationary equilibrium quantities such as the average energy density or magnetization that display critical behavior only at  $T = 0$ .

Traditionally, the vicinity of the critical point of equilibrium QPTs at finite temperatures has been characterized by spatio-temporal order parameter correlation functions, which in particular for the quantum Ising chain are known to display a relatively complex behavior [1]. We find that the stationary non-equilibrium current through the critical system itself may serve as a powerful tool in order to reveal quantum critical properties (which does of course not exclude the possibility of much more complex features to be found in, e.g., temporal non-equilibrium correlations such as current noise spectra).

We also find that neither the position nor the order of the phase transition are changed by adding dissipation, provided the Lindblad operators are evaluated at the correct frequencies, i.e., in positivity-conserving secular approximation. Our conclusion from this observation is that phenomenological extensions of QPTs which simply add constant dissipation terms on top of equilibrium models have to be treated with caution.

*Model.* — Our full model Hamiltonian  $\mathcal{H} = \mathcal{H}_S + \mathcal{H}_{SB} + \mathcal{H}_B$  consists of the quantum Ising chain in a transverse

field for  $N$  spins

$$\mathcal{H}_S = -g \sum_{i=1}^N \sigma_i^x - J \sum_{i=1}^N \sigma_i^z \sigma_{i+1}^z, \quad (1)$$

where  $g$  describes the coupling to an external magnetic field,  $J$  the inter-chain coupling to nearest neighbors, and periodic boundary conditions are assumed  $\sigma_{N+1}^z \equiv \sigma_1^z$ . The model is analytically diagonalizable for finite  $N$  and displays a second order quantum phase transition between a paramagnetic phase (for  $g > J$ ) and a ferromagnetic one ( $g < J$ ) [26]. Furthermore, transport through the chain is enabled by coupling to two equilibrium reservoirs  $\mathcal{H}_B = \sum_{\alpha=S,D} \sum_q \omega_{\alpha q} b_{\alpha q}^\dagger b_{\alpha q}$ , source (S) and drain (D), at different temperatures. Here, we consider non-interacting bosons (e.g., photons or phonons) with momentum  $q$  and energy  $\omega_{\alpha q}$  (we set  $\hbar = 1$ ), having in mind a thermal-transport type flow of energy that is mediated by boson-induced spin-flips via the interaction

$$\mathcal{H}_{SB} = \sum_{\alpha,i,q} \sigma_i^x [h_{\alpha q}^i b_{\alpha q}^\dagger + h_{\alpha q}^{i*} b_{\alpha q}] \quad (2)$$

with interaction constants  $h_{\alpha q}^i$  that depend on the specific realization of the model in an experimental context. We mention that the derivations below can also be carried out for fermionic reservoirs albeit with some modifications due to finite chemical potentials that occur in that case. More importantly, we choose a particularly simple case of  $\mathcal{H}_{SB}$  that is amenable for analytical calculation by assuming the coupling strengths as site-independent  $h_{\alpha q}^i \rightarrow h_{\alpha q}$ . Effectively, the coupling to the Ising chain is then via the large spin operator  $M_x \equiv \sum_i \sigma_i^x$ .

We first diagonalize  $\mathcal{H}_S$  in the usual way, applying a Jordan-Wigner, Fourier, and Bogoliubov transformation ([26, 27]) which, in the subspace of an even number of quasi-particles and for even  $N$  (these constraints will be tacitly assumed further-on), transforms the system Hamiltonian into  $\mathcal{H}_S^+ = \sum_k \epsilon_k (\gamma_k^\dagger \gamma_k - 1/2)$  with fermionic annihilation operators  $\gamma_k$  [27]. The quasi-momentum  $k$  may assume half-integer values  $k = \pm 1/2, \pm 3/2, \dots, \pm(N-1)/2$ . The single-particle energies are defined by

$$\epsilon_k = 2\Omega \sqrt{(1-s)^2 + s^2 - 2s(1-s) \cos\left(\frac{2\pi k}{N}\right)}, \quad (3)$$

where we have introduced a dimensionless phase parameter by fixing  $\Omega s = J$  and  $\Omega(1-s) = g$  with energy scale  $\Omega$ , see Fig. 1.

The very same transformation maps the interaction Hamiltonian, Eq. (2), to [27]

$$\mathcal{H}_{SB} = \left\{ N\mathbb{1} - 2 \sum_k \left[ |v_k|^2 \mathbb{1} + (|u_k|^2 - |v_k|^2) \gamma_k^\dagger \gamma_k \right. \right. \quad (4) \\ \left. \left. + u_k v_{-k} (\gamma_{+k}^\dagger \gamma_{-k}^\dagger + \gamma_{-k} \gamma_{+k}) \right] \right\} \sum_{\alpha,q} [h_{\alpha q} b_{\alpha q}^\dagger + \text{h.c.}],$$

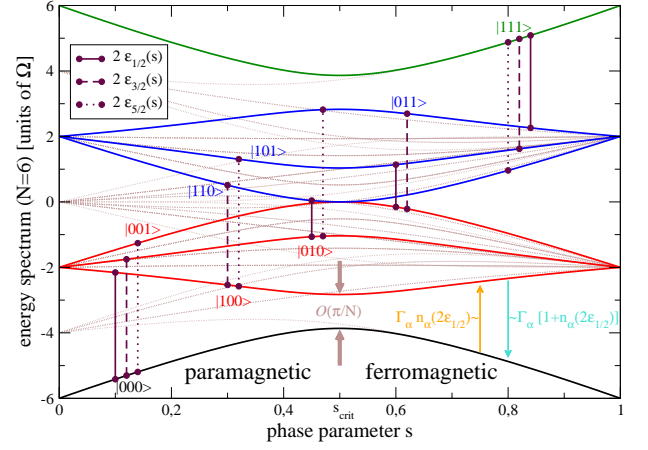


FIG. 1. (Color Online) Spectrum of the quantum Ising chain of length  $N = 6$  versus phase parameter  $s = J/\Omega = 1 - g/\Omega$  emphasizing the subspace of even quasi-particle numbers and vanishing pair momenta. In the thermodynamic limit  $N \rightarrow \infty$ , the second order quantum phase transition comes with a vanishing energy gap above the ground state, cf. Eq. (3). Possible transitions between states for second order perturbation in  $h_{\alpha k}$  are marked by vertical lines with corresponding single-pair energies  $2\epsilon_k$ . Thin dotted curves in the background complete the full spectrum of Eq. (1).

where the coefficients are defined by  $v_k \propto s \sin\left(\frac{2\pi k}{N}\right)$  and  $u_k \propto (1-s - s \cos\left(\frac{2\pi k}{N}\right) + \epsilon_k/(2\Omega))$  with normalization  $|u_k|^2 + |v_k|^2 = 1$ .

Obviously, the (spatially homogeneous) interaction  $\mathcal{H}_{SB}$  does not couple subspaces with different values of the total momentum  $P = \sum_k k \gamma_k^\dagger \gamma_k$ , since it either does not create particles at all (first line in Eq. (4)) or creates or annihilates only quasiparticle pairs of opposite momenta. In the paramagnetic phase with  $s \ll s_{\text{crit}}$ , these are spin waves running in opposite directions that can be interpreted as two hard-core bosons created or annihilated by the absorption or emission of a single reservoir photon/phonon. In the ferromagnetic domain  $s \gg s_{\text{crit}}$ , the quasi-particles correspond to kinks between domains of opposite spin directions.

Assuming that the system is initially prepared in the subspace of vanishing total momentum (e.g., in its ground state  $|0\rangle$ ), it now suffices to consider the subspace of pairs with opposite quasi-momenta only. In this subspace, the basis elements can be conveniently constructed from the ground state via

$$|\mathbf{n}\rangle = \left| n_{\frac{1}{2}}, n_{\frac{3}{2}}, \dots, n_{\frac{N-1}{2}} \right\rangle \equiv \prod_{k>0} (\gamma_{+k}^\dagger \gamma_{-k}^\dagger)^{n_k} |0\rangle, \quad (5)$$

where  $n_k \in \{0, 1\}$  denotes the occupation of a quasi-particle pair with momenta  $(+k, -k)$  such that  $(\gamma_k^\dagger \gamma_k + \gamma_{-k}^\dagger \gamma_{-k}) |\mathbf{n}\rangle = 2n_k |\mathbf{n}\rangle$ .

*Rate Equation.*— Applying lowest order perturbation theory in the coupling strength and in the relevant sub-

space (employing Born, Markov, and secular approximations [20] in the standard way appropriate for open systems) yields a rate equation

$$\dot{\rho}_{\mathbf{n}} = \sum_{\mathbf{m}} \left( \sum_{\alpha} \gamma_{\mathbf{n}\mathbf{m}}^{\alpha} \right) \rho_{\mathbf{m}} \quad (6)$$

for populations of the system energy eigenstates  $\rho_{\mathbf{n}} \equiv \langle \mathbf{n} | \rho | \mathbf{n} \rangle$ , where the transition rates  $\gamma_{\mathbf{n}\mathbf{m}}^{\alpha}$  due to reservoir  $\alpha$  admit only creation or annihilation of single quasi-particle pairs, see vertical lines in Fig. 1. Assuming thermal reservoir states, the transition rates ( $\mathbf{n} \neq \mathbf{m}$ ) evaluate to [27]  $\gamma_{\mathbf{n}\mathbf{m}}^{\alpha} = \Gamma_{\alpha}(\Delta\mathbf{m}\mathbf{n}) [1 + n_{\alpha}(\Delta\mathbf{m}\mathbf{n})] |\langle \mathbf{n} | M_x | \mathbf{m} \rangle|^2$  with energy differences  $\Delta\mathbf{m}\mathbf{n} \equiv E_{\mathbf{m}} - E_{\mathbf{n}}$ , system energies  $E_{\mathbf{n}} = \sum_{k>0} \epsilon_k (2n_k - 1)$ , spectral coupling densities  $\Gamma_{\alpha}(\omega) \equiv 2\pi \sum_q |h_{\alpha q}|^2 \delta(\omega - \omega_q)$ , and bosonic occupations  $n_{\alpha}(\omega) \equiv [e^{\beta_{\alpha}\omega} - 1]^{-1}$  (diagonal values  $\gamma_{\mathbf{n}\mathbf{n}}^{\alpha}$  follow from trace conservation).

When both reservoirs are held at sufficiently low temperatures such that  $n_{\alpha}(2\epsilon_k) \approx 0$  for all  $k \geq 3/2$ , the system will relax to the subspace spanned by the ground state  $|0\rangle \equiv |0, 0, \dots, 0\rangle$  and the first excited state  $|1\rangle \equiv |1, 0, \dots, 0\rangle$ . All states of higher occupations relax via the annihilation of higher-momentum quasi-particle pairs towards this subspace, see Fig. 1. The reduced transition rates are proportional to the matrix element [27]

$$|\langle 0 | M_x | 1 \rangle|^2 = \frac{s^2 \sin^2(\frac{\pi}{N})}{1 - 2s(1-s) [1 + \cos(\frac{\pi}{N})]} \quad (7)$$

with its second derivative with respect to  $s$  becoming nonanalytic at  $s_{\text{crit}} = 1/2$  as  $N \rightarrow \infty$ . In the continuum limit however, the dynamics cannot be constrained to the  $2 \times 2$  subspace anymore as many levels approach the ground state, cf. Eq. (3), which motivates study of the full-dimensional problem.

Using Eq. (6) and the rates  $\gamma_{\mathbf{n}\mathbf{m}}^{\alpha}$ , we obtain an analytical result for the steady state solution [27],

$$\bar{\rho}_{\mathbf{n}} = \prod_{k>0} \frac{[\bar{n}(2\epsilon_k)]^{n_k} [1 + \bar{n}(2\epsilon_k)]^{1-n_k}}{1 + 2\bar{n}(2\epsilon_k)}, \quad (8)$$

which – similar to Ref. [28] – is completely governed by an effective average bosonic occupation  $\bar{n}(\omega) \equiv \frac{\sum_{\alpha} \Gamma_{\alpha}(\omega) n_{\alpha}(\omega)}{\sum_{\alpha} \Gamma_{\alpha}(\omega)}$ . However, our system has more than one allowed transition frequency, which implies that the stationary state (8) is non-thermal (i.e., cannot be described by a single effective temperature) as soon as the reservoir temperatures are different ( $n_S(\omega) \neq n_D(\omega)$ ). Eq. (8) enables us to calculate the stationary values of the energy, the magnetization, and the current both for finite chain lengths and in the continuum limit  $N \rightarrow \infty$ .

*Energy and Magnetization.*— For the mean stationary

energy, we find [27]

$$\bar{E} = \sum_{k>0} \frac{-\epsilon_k}{1 + 2\bar{n}(2\epsilon_k)} \xrightarrow{N \rightarrow \infty} N \int_0^{1/2} \frac{-\epsilon(\kappa)}{1 + 2\bar{n}(2\epsilon(\kappa))} d\kappa, \quad (9)$$

where we have introduced the continuum of system energies  $\epsilon(\kappa) \equiv \epsilon_{(N\kappa)}$ . At zero temperature, where  $\bar{n}(2\epsilon(\kappa)) = 0$ , the system settles to the ground state, and the energy density can be expressed by a complete elliptic integral  $E/N \rightarrow -\frac{2\Omega}{\pi} \mathcal{E}_E(4s(1-s))$ , with a divergent second derivative at  $s_{\text{crit}} = 1/2$ . This divergence, which reflects the usual ground state QPT criticality of the Ising chain, is also predictable from analyzing the analytic structure of the integrand in (9) at zero temperature. For finite temperature and also in non-equilibrium setups where  $\bar{n}(\omega) > 0$  for all  $\omega > 0$ , the energy density and its higher derivatives remain analytic.

We find a similar behavior for the magnetization, which for large  $N$  becomes [27] ( $v(\kappa) \equiv v_{(N\kappa)}$ )

$$\langle M_x \rangle \rightarrow N \left[ 1 - 4 \int_0^{1/2} \frac{|v(\kappa)|^2 + \bar{n}(2\epsilon(\kappa))}{1 + 2\bar{n}(2\epsilon(\kappa))} d\kappa \right]. \quad (10)$$

At zero temperature, the integral is similarly solved by normal elliptic integrals and those of the first kind, which display a divergence in the first derivative of the magnetization density with respect to  $s$ . However, at finite temperature all derivatives of the magnetization density are analytic, which is most evident in the trivial high-temperature case where  $\bar{n}(2\epsilon(\kappa)) \rightarrow \infty$ .

*Heat Current.*— This changes drastically, however, when we consider the heat current through the Ising chain from one reservoir to the other. Analysis of the transition rates (e.g., by introducing energy counting fields as in [29]) yields our main result for the current of net emitted bosons at the drain,

$$I = \sum_{\mathbf{n}, \mathbf{m}} (E_{\mathbf{m}} - E_{\mathbf{n}}) \gamma_{\mathbf{n}\mathbf{m}}^D \bar{\rho}_{\mathbf{m}} \quad (11)$$

$$= \sum_{k>0} \frac{2\epsilon_k A_k^2 \Gamma_S(2\epsilon_k) \Gamma_D(2\epsilon_k) [n_S(2\epsilon_k) - n_D(2\epsilon_k)]}{\Gamma_S(2\epsilon_k) [1 + 2n_S(2\epsilon_k)] + \Gamma_D(2\epsilon_k) [1 + 2n_D(2\epsilon_k)]},$$

where the second line follows after a straightforward calculation by inserting the stationary state and explicitly evaluating the transition rates [27]. Here, we have introduced  $A_k \equiv 4u_k v_k = \frac{4s\Omega}{\epsilon_k} \sin(\frac{2\pi k}{N})$ .

Evidently, the current is antisymmetric when  $S \leftrightarrow D$ , vanishes at equilibrium, and is positive when the source temperature exceeds the drain temperature (which implies  $n_S(\omega) > n_D(\omega)$ ). Most important however, in the thermodynamic limit  $N \rightarrow \infty$  the current  $I$  directly reflects the signatures of the ground state quantum phase transition of the Ising chain. Formally, this correspondence is visible by the integral representation of  $I$ , which shows a divergence of its second derivative with respect

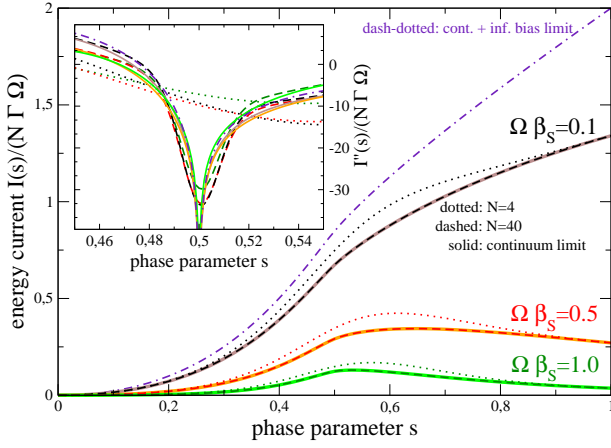


FIG. 2. (Color Online) Renormalized energy current  $I$  and its second derivative w.r.t.  $s$  (inset) versus phase parameter  $s$  for different chain lengths  $N = 4, 40, \infty$  (dotted, dashed, and bold solid, respectively) and for different source temperatures  $\Omega\beta_S = 0.1, 0.5, 1.0$  (black/brown, red/orange, and dark/light green, respectively). The dash-dotted purple curve denotes the analytically accessible case of  $n_D(\omega) \rightarrow 0$ ,  $n_S(\omega) \rightarrow \infty$ , and  $N \rightarrow \infty$ . Other parameters:  $\Omega\beta_D = 10, \Gamma_S(\epsilon_k) = \Gamma_D(\epsilon_k) = \Gamma$ .

to the phase parameter  $s$  at all temperatures, see Fig. 2. The second derivative of the integrand in the continuum version  $I/N \equiv \int_0^{1/2} j(s, \kappa) d\kappa$  of Eq. (11) will at the critical point  $s_{\text{crit}} = 1/2$  for small  $\kappa$  diverge as [27]

$$\left. \frac{\partial^2 j(s, \kappa)}{\partial^2 s} \right|_{s=1/2} \approx -\frac{32\Omega\Gamma_S\Gamma_D(\beta_D - \beta_S)}{\pi(\Gamma_S\beta_D + \Gamma_D\beta_S)\kappa} + \mathcal{O}\{\kappa\}, \quad (12)$$

whilst the integrand itself and its first derivative remain finite. Even for the extreme non-equilibrium, infinite thermobias regime the analytically obtainable [27] current displays a divergence of the second derivative at  $s_{\text{crit}} = 1/2$ , compare the dash-dotted curves in Fig. 2.

**Conclusion.**— For transport through a closed Ising chain homogeneously coupled to two bosonic thermal reservoirs we have analytically shown that signatures of the underlying QPT are manifest in the energy current already at finite temperature and also in the non-equilibrium regime – in contrast to system observables as mean energy or mean magnetization. For finite sizes  $N$ , all quantities remain analytic but precursors of the QPT are already visible at rather moderate chain lengths. Slightly perturbing the coupling symmetry, all subspaces of the model – cf. thin curves in Fig. 1 – may be weakly coupled, but since for near-multistable systems the separate current contributions are additive [30], we expect the sensitivity of the current to the underlying QPT to remain. Furthermore, in the weak-coupling limit considered here, the position of the phase transition is neither changed nor are new phases introduced by the coupling to reservoirs. We note that higher order transport

cumulants (noise or time-dependent current autocorrelation functions) can be expected to yield even more complex features and thus might prove useful tools to analyze QPTs out of equilibrium.

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## Supplementary Material

Equations in the main manuscript are referenced by square brackets.

### Mapping the system Hamiltonian to non-interacting fermions

The Jordan-Wigner transform (JWT)

$$\sigma_n^x = \mathbf{1} - 2c_n^\dagger c_n, \quad \sigma_n^z = -(c_n + c_n^\dagger) \prod_{m=1}^{n-1} (\mathbf{1} - 2c_m^\dagger c_m) \quad (1)$$

maps the spin-1/2 Pauli matrices non-locally to fermionic operators  $c_m$ . Inserting the JWT into the Hamiltonian [1] in the main manuscript, one has to treat the boundary term with special care

$$\begin{aligned} \mathcal{H}_S &= -g \sum_{n=1}^N (\mathbf{1} - 2c_n^\dagger c_n) - J \sum_{n=1}^{N-1} (c_n + c_n^\dagger)(c_{n+1} + c_{n+1}^\dagger)(\mathbf{1} - 2c_n^\dagger c_n) - J(c_N + c_N^\dagger) \left[ \prod_{n=1}^{N-1} (\mathbf{1} - 2c_n^\dagger c_n) \right] (c_1 + c_1^\dagger) \\ &= -g \sum_{n=1}^N (\mathbf{1} - 2c_n^\dagger c_n) - J \sum_{n=1}^{N-1} (c_n^\dagger - c_n)(c_{n+1}^\dagger + c_{n+1}) + J(c_N^\dagger - c_N)(c_1^\dagger + c_1) \left[ \prod_{n=1}^N (\mathbf{1} - 2c_n^\dagger c_n) \right], \end{aligned} \quad (2)$$

where we have extensively used the fermionic anticommutation relations. Introducing the projection operators on the subspaces with even (+) and odd (-) total number of fermion quasiparticles

$$\mathcal{P}^\pm \equiv \frac{1}{2} \left[ \mathbf{1} \pm \prod_{m=1}^N (\mathbf{1} - 2c_m^\dagger c_m) \right], \quad (3)$$

we may also write the Hamiltonian (2)  $\mathcal{H}_S = (\mathcal{P}^+ + \mathcal{P}^-)\mathcal{H}_S(\mathcal{P}^+ + \mathcal{P}^-)$ . It is straightforward to see that terms with different projectors and with  $n < N$  vanish

$$\begin{aligned} 0 &= \mathcal{P}^+(\mathbf{1} - 2c_n^\dagger c_n)\mathcal{P}^- = \mathcal{P}^-(\mathbf{1} - 2c_n^\dagger c_n)\mathcal{P}^+, \\ 0 &= \mathcal{P}^+(c_n^\dagger - c_n)(c_{n+1}^\dagger + c_{n+1})\mathcal{P}^- = \mathcal{P}^-(c_n^\dagger - c_n)(c_{n+1}^\dagger + c_{n+1})\mathcal{P}^+. \end{aligned} \quad (4)$$

For the boundary terms one finds similarly

$$\begin{aligned} &(\mathcal{P}^+ + \mathcal{P}^-)(c_N^\dagger - c_N)(c_1^\dagger + c_1) \left[ \prod_{n=1}^N (\mathbf{1} - 2c_n^\dagger c_n) \right] (\mathcal{P}^+ + \mathcal{P}^-) \\ &= (\mathcal{P}^+ + \mathcal{P}^-)(c_N^\dagger - c_N)(c_1^\dagger + c_1)(2\mathcal{P}^+ - \mathbf{1})(\mathcal{P}^+ + \mathcal{P}^-) \\ &= \mathcal{P}^+(c_N^\dagger - c_N)(c_1^\dagger + c_1)\mathcal{P}^+ - \mathcal{P}^-(c_N^\dagger - c_N)(c_1^\dagger + c_1)\mathcal{P}^-, \end{aligned} \quad (5)$$

such that we can finally write the Hamiltonian (2) as the sum of two non-interacting parts with either an even or an odd total number of fermionic quasiparticles

$$\begin{aligned} \mathcal{H}_S &= \mathcal{P}^+ \mathcal{H}_S^+ \mathcal{P}^+ + \mathcal{P}^- \mathcal{H}_S^- \mathcal{P}^- \\ &= \mathcal{P}^+ \left[ -g \sum_{n=1}^N (\mathbf{1} - 2c_n^\dagger c_n) - J \sum_{n=1}^N (c_n^\dagger - c_n)(c_{n+1}^\dagger + c_{n+1}) \right] \mathcal{P}^+ \\ &\quad + \mathcal{P}^- \left[ -g \sum_{n=1}^N (\mathbf{1} - 2c_n^\dagger c_n) - J \sum_{n=1}^N (c_n^\dagger - c_n)(c_{n+1}^\dagger + c_{n+1}) \right] \mathcal{P}^-. \end{aligned} \quad (6)$$

Note that this requires to define antiperiodic boundary conditions in the even (+) subspace  $c_{N+1}^{(+)} \equiv -c_1^{(+)}$  and periodic boundary conditions in the odd (-) subspace  $c_{N+1}^{(-)} \equiv +c_1^{(-)}$ .

Since the even subspace is relevant to our model, we further seek to diagonalize the Hamiltonian

$$\mathcal{H}_S^+ = -g \sum_{n=1}^N (\mathbf{1} - 2c_n^\dagger c_n) - J \sum_{n=1}^N (c_n^\dagger - c_n)(c_{n+1}^\dagger + c_{n+1}) \quad (7)$$

with antiperiodic boundary conditions  $c_{N+1} = -c_1$ . Translational invariance suggests to use the discrete Fourier transform (DFT, preserving the anticommutation relations due to its unitarity by construction)

$$c_n = \frac{e^{-i\pi/4}}{\sqrt{N}} \sum_k \tilde{c}_k e^{+ikn\frac{2\pi}{N}}, \quad (8)$$

which is compatible with the antiperiodic boundary conditions when  $N$  is even such that  $k$  takes half-integer values

$$k \in \{\pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}, \dots\}, \quad \text{where} \quad |k| \leq \frac{N-1}{2} \quad (9)$$

The DFT maps the Hamiltonian into

$$\mathcal{H}_S^+ = -gN\mathbf{1} + \sum_k \left\{ 2[g - J \cos(k2\pi/N)] \tilde{c}_k^\dagger \tilde{c}_k + J \sin(k2\pi/N) [\tilde{c}_k^\dagger \tilde{c}_{-k}^\dagger + \tilde{c}_{-k} \tilde{c}_k] \right\}. \quad (10)$$

Now, the observation that only positive and negative frequencies couple (conservation of one-dimensional quasi-momentum), suggests to use the reduced Bogoliubov transform

$$\tilde{c}_k = u_{+k} \gamma_{+k} + v_{-k}^* \gamma_{-k}^\dagger, \quad (11)$$

which mixes positive and negative momenta and where the a priori unknown coefficients have already been labeled suggestively (a more general ansatz would eventually of course yield the same solution). Since the new operators  $\gamma_k$  should be fermionic, we obtain from the orthonormality conditions

$$1 = |u_{+k}|^2 + |v_{-k}|^2, \quad 0 = u_{+k} v_{+k}^* + u_{-k} v_{-k}^* = (v_{+k}^*, v_{-k}^*) \begin{pmatrix} u_{+k} \\ u_{-k} \end{pmatrix}. \quad (12)$$

Demanding that the Bogoliubov transform eliminates all non-diagonal terms (of the form  $\gamma_{-k} \gamma_{+k}$  etc.) yields (by combining positive and negative  $k$ ) the equation

$$\begin{aligned} 0 &= 2 \left[ g - J \cos\left(k\frac{2\pi}{N}\right) \right] (u_{+k} v_{-k} - u_{-k} v_{+k}) + 2J \sin\left(k\frac{2\pi}{N}\right) (u_{-k} u_{+k} + v_{-k} v_{+k}) \\ &= (v_{-k}, u_{-k}) \begin{pmatrix} +2[g - J \cos(k\frac{2\pi}{N})] & +2J \sin(k\frac{2\pi}{N}) \\ +2J \sin(k\frac{2\pi}{N}) & -2[g - J \cos(k\frac{2\pi}{N})] \end{pmatrix} \begin{pmatrix} u_{+k} \\ v_{+k} \end{pmatrix} \equiv (v_{-k}, u_{-k}) \mathcal{M} \begin{pmatrix} u_{+k} \\ v_{+k} \end{pmatrix}. \end{aligned} \quad (13)$$

All equations can be fulfilled when we choose  $(u_{+k}, v_{+k})^T$  as the normalized positive energy eigenstate of the matrix  $\mathcal{M}$  with eigenvalue

$$\epsilon_k^+ = +2\sqrt{g^2 + J^2 - 2gJ \cos(k2\pi/N)} \equiv \epsilon_k \quad (14)$$

and  $(v_{-k}^*, u_{-k}^*)^T = (-v_{+k}^*, +u_{+k}^*)^T$  as its negative energy eigenstate with eigenvalue  $\epsilon_k^- = -2\sqrt{g^2 + J^2 - 2gJ \cos(k2\pi/N)}$ . To be more explicit, we have

$$\begin{aligned} u_k &= \frac{g - J \cos(k2\pi/N) + \sqrt{g^2 + J^2 - 2gJ \cos(k2\pi/N)}}{\sqrt{\left[g - J \cos(k2\pi/N) + \sqrt{g^2 + J^2 - 2gJ \cos(k2\pi/N)}\right]^2 + [J \sin(k2\pi/N)]^2}}, \\ v_k &= \frac{J \sin(k2\pi/N)}{\sqrt{\left[g - J \cos(k2\pi/N) + \sqrt{g^2 + J^2 - 2gJ \cos(k2\pi/N)}\right]^2 + [J \sin(k2\pi/N)]^2}}. \end{aligned} \quad (15)$$

Using these solutions, we obtain when  $N$  is even

$$\mathcal{H}_S^+ = \sum_k 2\sqrt{g^2 + J^2 - 2gJ \cos\left(k\frac{2\pi}{N}\right)} \left( \gamma_k^\dagger \gamma_k - \frac{1}{2} \right) = \sum_k \epsilon_k \left( \gamma_k^\dagger \gamma_k - \frac{1}{2} \right), \quad (16)$$

which reproduces the inline equation before [3] in the main manuscript. Note that the case of odd  $N$  and/or diagonalization in the odd subspace may be treated similarly.

### Mapping of the interaction Hamiltonian

Obviously, the used transformations do not affect the reservoir part, such that it suffices to transform  $M_x = \sum_{i=1}^N \sigma_i^x$  with the very same transformations as before. Inserting the JWT (1) yields

$$M_x = N\mathbf{1} - 2 \sum_{n=1}^N c_n^\dagger c_n. \quad (17)$$

Furthermore, inserting the DFT (8) leads to

$$M_x = N\mathbf{1} - 2 \sum_k \tilde{c}_k^\dagger \tilde{c}_k. \quad (18)$$

Finally, inserting the Bogoliubov transformation (11), replacing  $k \rightarrow -k$  in some terms and exploiting that the coefficients (15) are real yields

$$M_x = N\mathbf{1} - 2 \sum_k \left[ |u_k|^2 \gamma_k^\dagger \gamma_k + |v_k|^2 \gamma_k \gamma_k^\dagger + u_k v_{-k} \left( \gamma_{+k}^\dagger \gamma_{-k}^\dagger + \gamma_{-k} \gamma_{+k} \right) \right], \quad (19)$$

which by using the fermionic anticommutation relations is equivalent to Eq. [4] in the main manuscript. For later convenience we write this as a sum over positive  $k$ -values only

$$M_x = N\mathbf{1} - 4 \sum_{k>0} |v_k|^2 \mathbf{1} - 2 \sum_{k>0} \left( 1 - 2|v_k|^2 \right) \left( \gamma_k^\dagger \gamma_k + \gamma_{-k}^\dagger \gamma_{-k} \right) + 4 \sum_{k>0} u_k v_k \left( -\gamma_{+k}^\dagger \gamma_{-k}^\dagger + \gamma_{+k} \gamma_{-k} \right). \quad (20)$$

### Derivation of the rate equation

We rely on previous results in the literature that yield for an interaction Hamiltonian of the form  $\mathcal{H}_{\text{SB}} = A \otimes B$  under Born, Markov, and secular approximations a Lindblad-type master equation. When the spectrum of the system Hamiltonian is non-degenerate (and here more specifically, when states coupled in the master equation are energetically non-degenerate), this Lindblad master equation couples only the diagonal elements of the density matrix in the system energy eigenbasis to each other, i.e., it can be written in the form of a rate equation

$$\dot{\rho}_{aa} = \sum_b \gamma_{ab,ab} \rho_{bb} - \sum_b \gamma_{ba,ba} \rho_{aa}, \quad (21)$$

where  $a, b$  label the different system energy eigenstates. Note that the refined condition of non-degeneracy is for finite  $N$  always fulfilled, as e.g. the intersection point in Fig. 1 between  $|001\rangle$  and  $|110\rangle$  in the main manuscript are between uncoupled states. The transition rates are given by

$$\gamma_{ab,ab} = \gamma(E_b - E_a) |\langle a| A | b \rangle|^2, \quad (22)$$

where  $\gamma(\omega)$  is the Fourier transform of the bath correlation function,  $\gamma(\omega) \equiv \int C(\tau) e^{+i\omega\tau} d\tau \equiv \int \langle e^{+i\mathcal{H}_S\tau} B e^{-i\mathcal{H}_S\tau} B \rangle e^{+i\omega\tau} d\tau$ . Specifically for our model there is only one, which for two thermal source and drain reservoirs becomes

$$\begin{aligned} C(\tau) &= \sum_{\alpha\alpha'qq'} \left\langle \left[ h_{\alpha q} b_{\alpha q}^\dagger e^{+i\omega_{\alpha q}\tau} + \text{h.c.} \right] \left[ h_{\alpha' q'} b_{\alpha' q'}^\dagger + \text{h.c.} \right] \right\rangle = \sum_{\alpha} \sum_q |h_{\alpha q}|^2 \left[ \langle b_{\alpha q}^\dagger b_{\alpha q} \rangle e^{+i\omega_{\alpha q}\tau} + \langle b_{\alpha q} b_{\alpha q}^\dagger \rangle e^{-i\omega_{\alpha q}\tau} \right] \\ &= \frac{1}{2\pi} \sum_{\alpha} \int_0^{\infty} \Gamma_{\alpha}(\omega) \left[ n_{\alpha}(\omega) e^{+i\omega\tau} + (1 + n_{\alpha}(\omega)) e^{-i\omega\tau} \right] d\omega, \end{aligned} \quad (23)$$

where we have introduced for  $\omega > 0$  the spectral coupling density  $\Gamma_{\alpha}(\omega) \equiv 2\pi \sum_q |h_{\alpha q}|^2 \delta(\omega - \omega_{\alpha q})$  (recall that  $\omega_{\alpha q} > 0$ ) and  $n_{\alpha}(\omega) \equiv [e^{\beta_{\alpha}\omega} - 1]^{-1}$  denotes the Bose distribution for reservoir  $\alpha$  held at inverse temperature  $\beta_{\alpha}$ . Analytically continuing the spectral coupling densities to negative frequencies via  $\Gamma_{\alpha}(-\omega) \equiv -\Gamma_{\alpha}(\omega)$  and exploiting that  $n_{\alpha}(-\omega) = -(1 + n_{\alpha}(\omega))$  yields after a simple integral transformation

$$C(\tau) = \frac{1}{2\pi} \sum_{\alpha} \int_{-\infty}^{+\infty} \Gamma_{\alpha}(\omega) [1 + n_{\alpha}(\omega)] e^{+i\omega\tau} d\omega, \quad (24)$$

which enables to directly read off the Fourier transform  $\gamma(\omega) = \sum_{\alpha} \Gamma_{\alpha}(\omega) [1 + n_{\alpha}(\omega)] \equiv \sum_{\alpha} \gamma_{\alpha}(\omega)$ . Evidently, the contributions of the two reservoirs enter additively in the rate equations, such that by labeling the energy eigenstates in the relevant subspace ( $a \rightarrow \mathbf{n}$ ) we recover the rate equation with its coefficients stated below Eq. [6] in the main manuscript.

### Low Temperature Limit

At sufficiently low temperatures, such that  $n_{\alpha}(2\epsilon_k) \ll 1$  for all  $k \geq 3/2$  but still  $n_{\alpha}(2\epsilon_{1/2}) = \mathcal{O}(1)$ , it is evident from Fig. 1 that most excited states will relax towards the two lowest states  $|0\rangle \equiv |0, 0 \dots 0\rangle$  and  $|1, 0 \dots 0\rangle \equiv |1\rangle$ . The dynamics in this subspace is governed by the rate equation

$$\begin{pmatrix} \dot{\rho}_0 \\ \dot{\rho}_1 \end{pmatrix} = |\langle 0 | M_x | 1 \rangle|^2 \sum_{\alpha} \Gamma_{\alpha}(2\epsilon_{1/2}) \begin{pmatrix} -n_{\alpha}(2\epsilon_{1/2}) & + [1 + n_{\alpha}(2\epsilon_{1/2})] \\ +n_{\alpha}(2\epsilon_{1/2}) & - [1 + n_{\alpha}(2\epsilon_{1/2})] \end{pmatrix} \begin{pmatrix} \rho_0 \\ \rho_1 \end{pmatrix} \quad (25)$$

with the matrix element

$$|\langle 0 | M_x | 1 \rangle|^2 = 4u_{1/2}^2 v_{1/2}^2 = \frac{s^2 \sin^2\left(\frac{\pi}{N}\right)}{1 - 2s(1-s) \left[1 + \cos\left(\frac{\pi}{N}\right)\right]}, \quad (26)$$

compare Eq. [7] in the main text. Consequently, the current in this effective low-temperature limit becomes

$$I = \frac{\Gamma_S(2\epsilon_{1/2})\Gamma_D(2\epsilon_{1/2}) [n_S(2\epsilon_{1/2}) - n_D(2\epsilon_{1/2})]}{\Gamma_S(2\epsilon_{1/2}) [1 + 2n_S(2\epsilon_{1/2})] + \Gamma_D(2\epsilon_{1/2}) [1 + 2n_D(2\epsilon_{1/2})]} \frac{s^2 \sin^2\left(\frac{\pi}{N}\right)}{1 - 2s(1-s) \left[1 + \cos\left(\frac{\pi}{N}\right)\right]}, \quad (27)$$

which modifies the usual bosonic current through a two-level system by the matrix element  $|\langle 0 | M_x | 1 \rangle|^2$ . Eventually, the  $s$ -dependence of this prefactor yields the non-monotonous dependence of the current on the phase-parameter  $s$  in the low-temperature curves in Fig. 2.

### Non-equilibrium Stationary State

The stationary solution of the rate equation can even for non-equilibrium (different temperature) configurations be obtained using basically two ingredients. First, we note that the Fourier transforms of the bath correlation functions obey the usual Kubo-Martin-Schwinger conditions  $\gamma_{\alpha}(-\omega) = e^{-\beta_{\alpha}\omega} \gamma_{\alpha}(+\omega)$ , which lead when the system is coupled to only one bath (e.g. by setting  $\Gamma_D(\omega) = 0$ ) to thermalization of the system with the temperature of the remaining reservoir (e.g.  $\beta_S^{-1}$ ). Formally, such a thermal state is characterized by the ratio of diagonal elements to be

$$\frac{\bar{\rho}_{\mathbf{n}}}{\bar{\rho}_{\mathbf{m}}} = e^{-\beta(E_{\mathbf{n}} - E_{\mathbf{m}})} = \frac{n(E_{\mathbf{n}} - E_{\mathbf{m}})}{1 + n(E_{\mathbf{n}} - E_{\mathbf{m}})}, \quad (28)$$

where  $n(\omega)$  corresponds to the Bose distribution of the connected reservoir. For coupling to multiple reservoirs we use that the occupations of the different reservoirs enter linearly and just weighted by the different tunneling rates to motivate the ansatz ( $\Delta_{nm} \equiv E_{\mathbf{n}} - E_{\mathbf{m}}$ )

$$\frac{\bar{\rho}_{\mathbf{n}}}{\bar{\rho}_{\mathbf{m}}} = \frac{\bar{n}(\Delta_{nm})}{1 + \bar{n}(\Delta_{nm})}, \quad \bar{n}(\omega) \equiv \frac{\Gamma_S(\omega)n_S(\omega) + \Gamma_D(\omega)n_D(\omega)}{\Gamma_S(\omega) + \Gamma_D(\omega)}. \quad (29)$$

Indeed, one can easily prove for the rate equation

$$\begin{aligned} \dot{\rho}_{\mathbf{n}} = & \sum_{\mathbf{m} \neq \mathbf{n}} \sum_{\alpha} \Gamma_{\alpha}(\Delta_{mn}) [1 + n_{\alpha}(\Delta_{mn})] |\langle \mathbf{n} | M_x | \mathbf{m} \rangle|^2 \rho_{\mathbf{m}} \\ & - \left( \sum_{\mathbf{m} \neq \mathbf{n}} \sum_{\alpha} \Gamma_{\alpha}(\Delta_{nm}) [1 + n_{\alpha}(\Delta_{nm})] |\langle \mathbf{m} | M_x | \mathbf{n} \rangle|^2 \right) \rho_{\mathbf{n}} \end{aligned} \quad (30)$$

the validity of the stationary state by inserting

$$\bar{\rho}_{\mathbf{m}} = \frac{\bar{n}(\Delta_{mn})}{1 + \bar{n}(\Delta_{mn})} \bar{\rho}_{\mathbf{n}} = \frac{\sum_{\alpha} \Gamma_{\alpha}(\Delta_{mn}) n_{\alpha}(\Delta_{mn})}{\sum_{\alpha} \Gamma_{\alpha}(\Delta_{mn}) [1 + n_{\alpha}(\Delta_{mn})]} \bar{\rho}_{\mathbf{n}} \quad (31)$$



and using that  $\Gamma_\alpha(\Delta \mathbf{m} \mathbf{n}) = -\Gamma_\alpha(\Delta \mathbf{n} \mathbf{m})$  and  $n_\alpha(\Delta \mathbf{m} \mathbf{n}) = -[1 + n_\alpha(\Delta \mathbf{n} \mathbf{m})]$ . By a sequence of pair annihilations – compare Fig. 1 in the main manuscript – it therefore follows that any stationary occupation may be connected to the ground state occupation  $\bar{\rho}_0$  via

$$\bar{\rho} \mathbf{n} = \bar{\rho}_0 \prod_{k>0} \left( \frac{\bar{n}(2\epsilon_k)}{1 + \bar{n}(2\epsilon_k)} \right)^{n_k}. \quad (32)$$

The latter is fixed by the normalization  $\text{Tr} \{ \bar{\rho} \mathbf{n} \} = 1$

$$1 = \bar{\rho}_0 \sum_{n_{1/2}=0}^1 \dots \sum_{n_{(N-1)/2}=0}^1 \prod_{k>0} \left( \frac{\bar{n}(2\epsilon_k)}{1 + \bar{n}(2\epsilon_k)} \right)^{n_k} = \bar{\rho}_0 \prod_{k>0} \left[ \sum_{n_k=0}^1 \left( \frac{\bar{n}(2\epsilon_k)}{1 + \bar{n}(2\epsilon_k)} \right)^{n_k} \right] = \bar{\rho}_0 \prod_{k>0} \frac{1 + 2\bar{n}(2\epsilon_k)}{1 + \bar{n}(2\epsilon_k)} \quad (33)$$

which yields Eq. [8] in the manuscript.

### Stationary Energy

Rewriting the system Hamiltonian (16) as

$$\mathcal{H}_S^+ = \sum_{k>0} \epsilon_k \left( \gamma_k^\dagger \gamma_k + \gamma_{-k}^\dagger \gamma_{-k} - 1 \right) \quad (34)$$

implies for its diagonal matrix elements in the relevant subspace  $\langle \mathbf{n} | \mathcal{H}_S^+ | \mathbf{n} \rangle = \sum_{k>0} \epsilon_k (2n_k - 1)$ . The stationary expectation value of the system energy then becomes with Eq. [8] in the main text

$$\begin{aligned} \langle \bar{E} \rangle &= \text{Tr} \{ \mathcal{H}_S^+ \bar{\rho} \} = \sum_{\mathbf{n}} \langle \mathbf{n} | \mathcal{H}_S^+ | \mathbf{n} \rangle \bar{\rho} \mathbf{n} = \sum_{k>0} \epsilon_k \sum_{\mathbf{n}} (2n_k - 1) \bar{\rho} \mathbf{n} \\ &= \sum_{k>0} \epsilon_k \sum_{n_k=0}^1 \frac{[\bar{n}(2\epsilon_k)]^{n_k} [1 + \bar{n}(2\epsilon_k)]^{1-n_k}}{1 + 2\bar{n}(2\epsilon_k)} (2n_k - 1) = \sum_{k>0} \frac{-\epsilon_k}{1 + 2\bar{n}(2\epsilon_k)}, \end{aligned} \quad (35)$$

where we have used that  $\sum_{n_k=0}^1 \frac{\bar{n}^{n_k} [1 + \bar{n}]^{1-n_k}}{1 + 2\bar{n}} = 1$  holds for each  $k$  separately in the second line. In the thermodynamic limit ( $N \rightarrow \infty$ ) and noting that all relevant quantities actually depend on  $\kappa = k/N$ , the sum is easily converted into an integral, and we recover Eq. [9] in the main text.

### Stationary Magnetization

Similarly, we evaluate the diagonal matrix elements of the magnetization operator (20)

$$\langle \mathbf{n} | M_x | \mathbf{n} \rangle = N - 4 \sum_{k>0} |v_k|^2 - 4 \sum_{k>0} (|u_k|^2 - |v_k|^2) n_k = N - 4 \sum_{k>0} \left[ |v_k|^2 + (1 - 2|v_k|^2) n_k \right], \quad (36)$$

which can be inserted in the stationary expectation value to yield

$$\begin{aligned} \langle \bar{M}_x \rangle &= \sum_{\mathbf{n}} \langle \mathbf{n} | M_x | \mathbf{n} \rangle \bar{\rho} \mathbf{n} = N - 4 \sum_{k>0} |v_k|^2 - 4 \sum_{k>0} (1 - 2|v_k|^2) \sum_{n_k=0}^1 n_k \frac{[\bar{n}(2\epsilon_k)]^{n_k} [1 + \bar{n}(2\epsilon_k)]^{1-n_k}}{1 + 2\bar{n}(2\epsilon_k)} \\ &= N - 4 \sum_{k>0} |v_k|^2 - 4 \sum_{k>0} (1 - 2|v_k|^2) \frac{\bar{n}(2\epsilon_k)}{1 + 2\bar{n}(2\epsilon_k)} = N - 4 \sum_{k>0} \frac{|v_k|^2 + \bar{n}(2\epsilon_k)}{1 + 2\bar{n}(2\epsilon_k)}. \end{aligned} \quad (37)$$

Finally, the sum over  $k$  can similarly be converted into an integral to yield Eq. [10] in the main text. Furthermore, by inserting the coefficient (15) in the continuum representation and zero-temperature limit, we obtain for the magnetization density

$$\langle \bar{m}_x \rangle = \frac{\langle \bar{M}_x \rangle}{N} = 1 - 4 \int_0^{1/2} v^2(\kappa) d\kappa = \frac{\mathcal{E}_E(4s(1-s)) + (1-2s)\mathcal{E}_K(4s(1-s))}{\pi(1-s)}, \quad (38)$$

where  $\mathcal{E}_E(x)$  and  $\mathcal{E}_K(x)$  denote the complete elliptic integral and the complete elliptic integral of the first kind, respectively.

### Stationary Current

The stationary current of bosons emitted to the drain can for example be obtained by inserting energy counting fields in the off-diagonal matrix elements of the rate equation matrix, i.e., to perform in Eq. (30) the replacements

$$\Gamma_D(\Delta \mathbf{m} \mathbf{n}) [1 + n_D(\Delta \mathbf{m} \mathbf{n})] \rightarrow \Gamma_D(\Delta \mathbf{m} \mathbf{n}) [1 + n_D(\Delta \mathbf{m} \mathbf{n})] e^{+i\Delta \mathbf{m} \mathbf{n} \chi}, \quad (39)$$

which automatically takes into account that  $\Delta \mathbf{m} \mathbf{n} > 0$  corresponds to emission into the drain and  $\Delta \mathbf{m} \mathbf{n} < 0$  to absorption. Note that in the latter case one would use  $\Gamma_D(-x) [1 + n_D(-x)] = \Gamma_D(+x) n_D(+x)$ . This upgrades the rate equation by a counting field  $\dot{\rho} = \mathcal{L}(\chi)\rho$ , and the stationary current can then be obtained from the stationary state by deriving the rate matrix with respect to the counting field  $\chi$

$$\begin{aligned} I &= (-i) \text{Tr} \{ \mathcal{L}'(0) \bar{\rho} \} = \sum_{\mathbf{n}} \sum_{\mathbf{m} \neq \mathbf{n}} \Delta \mathbf{m} \mathbf{n} \Gamma_D(\Delta \mathbf{m} \mathbf{n}) [1 + n_D(\Delta \mathbf{m} \mathbf{n})] |\langle \mathbf{n} | M_x | \mathbf{m} \rangle|^2 \bar{\rho}_{\mathbf{m}} \\ &= \sum_{\mathbf{n} \mathbf{m} : \Delta \mathbf{m} \mathbf{n} > 0} \Delta \mathbf{m} \mathbf{n} \Gamma_D(\Delta \mathbf{m} \mathbf{n}) [1 + n_D(\Delta \mathbf{m} \mathbf{n})] |\langle \mathbf{n} | M_x | \mathbf{m} \rangle|^2 \bar{\rho}_{\mathbf{m}} \\ &\quad - \sum_{\mathbf{n} \mathbf{m} : \Delta \mathbf{n} \mathbf{m} > 0} \Delta \mathbf{n} \mathbf{m} \Gamma_D(\Delta \mathbf{n} \mathbf{m}) n_D(\Delta \mathbf{n} \mathbf{m}) |\langle \mathbf{n} | M_x | \mathbf{m} \rangle|^2 \bar{\rho}_{\mathbf{m}} \\ &= \sum_{\mathbf{m}} \sum_{k>0} \left[ 2\epsilon_k m_k \Gamma_D(2\epsilon_k) [1 + n_D(2\epsilon_k)] (4u_k v_k)^2 \bar{\rho}_{\mathbf{m}} - 2\epsilon_k (1 - m_k) \Gamma_D(2\epsilon_k) n_D(2\epsilon_k) (4u_k v_k)^2 \bar{\rho}_{\mathbf{m}} \right] \\ &= \sum_{k>0} 2\epsilon_k \Gamma_D(2\epsilon_k) (4u_k v_k)^2 \sum_{\mathbf{m}} [m_k [1 + n_D(2\epsilon_k)] - (1 - m_k) n_D(2\epsilon_k)] \bar{\rho}_{\mathbf{m}} \\ &= \sum_{k>0} 2\epsilon_k \Gamma_D(2\epsilon_k) (4u_k v_k)^2 \sum_{m_k=0}^1 [m_k [1 + n_D(2\epsilon_k)] - (1 - m_k) n_D(2\epsilon_k)] \frac{[\bar{n}(2\epsilon_k)]^{m_k} [1 + \bar{n}(2\epsilon_k)]^{1-m_k}}{1 + 2\bar{n}(2\epsilon_k)} \\ &= \sum_{k>0} 2\epsilon_k \Gamma_D(2\epsilon_k) (4u_k v_k)^2 \frac{\bar{n}(2\epsilon_k) - n_D(2\epsilon_k)}{1 + 2\bar{n}(2\epsilon_k)} \\ &= \sum_{k>0} 2\epsilon_k (4u_k v_k)^2 \frac{\Gamma_S(2\epsilon_k) \Gamma_D(2\epsilon_k) [n_S(2\epsilon_k) - n_D(2\epsilon_k)]}{\Gamma_S(2\epsilon_k) [1 + 2n_S(2\epsilon_k)] + \Gamma_D(2\epsilon_k) [1 + 2n_D(2\epsilon_k)]}, \end{aligned} \quad (40)$$

which with evaluating the prefactor  $A_k \equiv 4u_k v_k$  from (15) becomes Eq. [11] in the main manuscript.

The continuum representation of the current becomes (in wide-band limit  $\Gamma_\alpha \equiv \Gamma_\alpha(2\epsilon_k)$ )

$$\frac{I}{N} = 32 \int_0^{1/2} \frac{s^2 \Omega^2 \sin^2(2\pi\kappa)}{\epsilon(\kappa)} \frac{\Gamma_S \Gamma_D [n_S(2\epsilon(\kappa)) - n_D(2\epsilon(\kappa))]}{\Gamma_S [1 + 2n_S(2\epsilon(\kappa))] + \Gamma_D [1 + 2n_D(2\epsilon(\kappa))]} d\kappa \equiv \int_0^{1/2} j(s, \kappa) d\kappa. \quad (41)$$

At the critical point and for small  $\kappa$ , the integrand behaves as

$$\begin{aligned} j(1/2, \kappa) &= \frac{8\pi\Omega(\beta_D - \beta_S)\Gamma_D\Gamma_S}{\Gamma_S\beta_D + \Gamma_D\beta_S} \kappa + \mathcal{O}\{\kappa^2\}, \\ \frac{\partial}{\partial s} j(s, \kappa) \Big|_{s=1/2} &= \frac{32\pi\Omega(\beta_D - \beta_S)\Gamma_D\Gamma_S}{\Gamma_S\beta_D + \Gamma_D\beta_S} \kappa + \mathcal{O}\{\kappa^2\}, \end{aligned} \quad (42)$$

which together with Eq. [12] in the main manuscript leads to divergence of the second derivative of the current at the critical point for all temperature configurations.

This can also be seen in closed form in the infinite thermobias regime ( $n_S(2\epsilon(\kappa)) \rightarrow \infty$  and  $n_D(2\epsilon(\kappa)) \rightarrow 0$ ), where (41) becomes

$$\begin{aligned} \frac{I}{N} &\rightarrow 16\Gamma_D(s\Omega)^2 \int_0^{1/2} \frac{\sin^2(2\pi\kappa)}{\epsilon(\kappa)} d\kappa \\ &= \frac{4\Gamma_D\Omega}{3\pi(1-s)^2} [(1-2s(1-s))\mathcal{E}_E(4s(1-s)) - (1-2s)^2\mathcal{E}_K(4s(1-s))], \end{aligned} \quad (43)$$

where  $\mathcal{E}_E(x)$  represents the complete elliptic integral and  $\mathcal{E}_K(x)$  the complete elliptic integral of the first kind.